6. Max-product algorithm

- MAP elimination algorithm
- Max-product algorithm on a tree
- Example: LDPC (Low-Density Parity Check) code over BSC (Binary Symmetric Channel)

Inference tasks on graphical models

consider an undirected graphical model (a.k.a. Markov random field)

$$\mu(x) \quad = \quad \frac{1}{Z} \prod_{c \in \mathcal{C}} \psi_c(x_c)$$

where $\mathcal{C}$ is the set of all maximal cliques in G

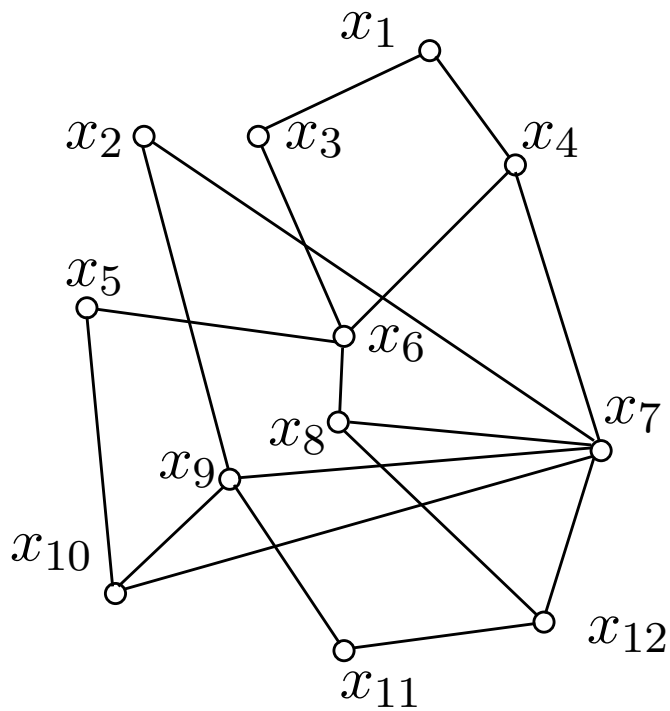
we want to

- calculate marginals: $\mu(x_A) = \sum_{x_{V \setminus A}} \mu(x)$
- calculating conditional distributions

$$\mu(x_A | x_B) \quad = \quad \frac{\mu(x_A, x_B)}{\mu(x_B)}$$

- calculation maximum a posteriori estimates: $\arg\max_{\hat{x}} \mu(\hat{x})$
- calculating the partition function Z
- sample from this distribution

MAP estimation = Mode computation



MAP estimation of x given y

$$x^* \in \arg\max_{x \in \mathcal{X}^V} \mu(x|y)$$

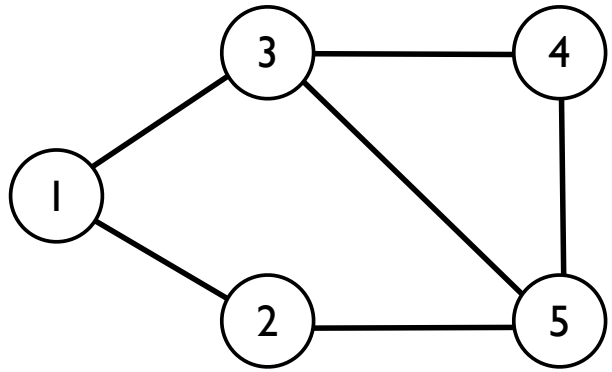
Mode computation

$$x^* \in \arg\max_{x \in \mathcal{X}^V} \mu(x)$$

$$= \arg\max_{x \in \mathcal{X}^V} \prod_{c \in \mathcal{C}} \psi_c(x_c)$$

MAP elimination algorithm for computing the mode

MAP elimination algorithm is exact but can require $O(|\mathcal{X}|^{|V|})$ operations



$$\mu(x) \quad \propto \quad e^{x_1 x_2} e^{-x_1 x_3} e^{x_2 x_5} e^{-x_3 x_4 x_5}$$

- $x_i \in \{+1, -1\}$
- we want to find $x^* \in \arg\max_x \mu(x)$
- brute force combinatorial search:

$$\max_{x \in \{+1,-1\}^5} \mu(x) \propto \max_{x_1,x_2,x_3,x_4,x_5} e^{x_1 x_2} e^{-x_1 x_3} e^{x_2 x_5} e^{-x_3 x_4 x_5}$$

- requires $O(|\mathcal{C}| \cdot |\mathcal{X}|^5)$ operations

- consider an elimination ordering $(5, 4, 3, 2, 1)$

$$
\begin{aligned}
\max_x \mu(x) \quad &\propto \quad \max_{x_1,x_2,x_3,x_4,x_5} e^{x_1x_2} e^{-x_1x_3} e^{x_2x_5} e^{-x_3x_4x_5} \\
&= \quad \max_{x_1,x_2,x_3,x_4} e^{x_1x_2} e^{-x_1x_3} \underbrace{\max_{x_5 \in \mathcal{X}} e^{(x_2 - x_3x_4)x_5}}_{\equiv m_5(x_2,x_3,x_4) = e^{|x_2 - x_3x_4|}, \text{ with } x_5^* = \text{sign}(x_2 - x_3x_4)} \\
&= \quad \max_{x_1,x_2,x_3,x_4} e^{x_1x_2} e^{-x_1x_3} m_5(x_2, x_3, x_4) \\
&= \quad \max_{x_1,x_2,x_3} e^{x_1x_2} e^{-x_1x_3} \underbrace{\max_{x_4 \in \mathcal{X}} m_5(x_2, x_3, x_4)}_{\equiv m_4(x_2,x_3) = e^2, \text{ with } x_4^* = -(x_2x_3)} \\
&= \quad \max_{x_1,x_2,x_3} e^{x_1x_2} e^{-x_1x_3} m_4(x_2, x_3) \\
&= \quad \max_{x_1,x_2} e^{x_1x_2} \underbrace{\max_{x_3 \in \mathcal{X}} e^{-x_1x_3} m_4(x_2, x_3)}_{\equiv m_3(x_1,x_2) = e^3, \text{ with } x_3^* = -x_1} \\
&= \quad \max_{x_1,x_2} e^{x_1x_2} m_3(x_1, x_2)
\end{aligned}
$$

in the final step, let $(x_1^*, x_2^*) \in \arg\max e^{3+x_1x_2}$, that is $x_2^* = x_1$ and $x_1^* = +1$ or -1

- from
  $x_1^* \in \{\pm 1\}, x_2^* = x_1, x_3^* = -x_1, x_4^* = -x_2 x_3, x_5^* = \mathrm{sign}(x_2 - x_3 x_4)$,
  we can find a MAP assignment by backtracking $(5, 4, 3, 2, 1)$:

$$x^* = \big(+1, +1, -1, +1, +1 \big) \text{ or } \big(-1, -1, +1, +1, -1 \big)$$

  maximizes $e^{x_1 x_2} e^{-x_1 x_3} e^{x_2 x_5} e^{-x_3 x_4 x_5}$
- since we are only searching for one maximizing configuration, we are free to break ties arbitrarily
- computational complexity depends on the elimination ordering
- how do we know which ordering is better?

Computational complexity of elimination algorithm

$$\mu(x_1) \propto \max_{x_2,x_3,x_4,x_5 \in \mathcal{X}} e^{x_1x_2} e^{-x_1x_3} e^{x_2x_5} e^{-x_3x_4x_5}$$

$$= \max_{x_2,x_3,x_4 \in \mathcal{X}} e^{x_1x_2} e^{-x_1x_3} \cdot \underbrace{\max_{x_5 \in \mathcal{X}} e^{x_2x_5} e^{-x_3x_4x_5}}_{\equiv m_5(S_5),\, S_5=\{x_2,x_3,x_4\},\, \Psi_5=\{e^{x_2x_5}, e^{-x_3x_4x_5}\}}$$

$$= \max_{x_2,x_3 \in \mathcal{X}} e^{x_1x_2} e^{-x_1x_3} \cdot \underbrace{\max_{x_4 \in \mathcal{X}} m_5(x_2,x_3,x_4)}_{\equiv m_4(S_4),\, S_4=\{x_2,x_3\},\, \Psi_4=\{m_5\}}$$

$$= \max_{x_2 \in \mathcal{X}} e^{x_1x_2} \cdot \underbrace{\max_{x_3 \in \mathcal{X}} e^{-x_1x_3} m_4(x_2,x_3)}_{\equiv m_3(S_3),\, S_3=\{x_1,x_2\},\, \Psi_3=\{e^{-x_1x_3}, m_4\}}$$

$$= \underbrace{\max_{x_2 \in \mathcal{X}} e^{x_1x_2} m_3(x_1,x_2)}_{\equiv m_2(S_2),\, S_2=\{x_1\},\, \Psi_2=\{e^{x_1x_2}, m_3\}} = m_2(x_1)$$

- total complexity: $\sum_i O(|\Psi_i| \cdot |\mathcal{X}|^{1+|S_i|}) = O(|V| \cdot \max_i |\Psi_i| \cdot |\mathcal{X}|^{1+max_i|S_i|})$
- use induced graph for graph theoretic complexity analysis

MAP elimination algorithm

- **input**: $\{\psi_c\}_{c \in \mathcal{C}}$, alphabet $\mathcal{X}$, elimination ordering I
- **output**: $x^* \in \arg\max \mu(x)$

1. initialize active set Ψ to be the set of input compatibility functions
2. **for** node i in I that is not in A **do**

   let S_i be the set of nodes, not including i, that share a compatibility function with i

   let Ψ_i be the set of compatibility functions in Ψ involving x_i

   compute

$$m_i(x_{S_i}) = \max_{x_i} \prod_{\psi \in \Psi_i} \psi(x_i, x_{S_i})$$

$$x_i^*(x_{S_i}) \in \arg\max_{x_i} \prod_{\psi \in \Psi_i} \psi(x_i, x_{S_i})$$

   remove elements of Ψ_i from Ψ

   add m_i to Ψ

   end
3. produce x^* by traversing I in the reverse order

$$x^*_{I(j)} = x^*_{I(j)}(x^*_{I(k)} : j < k)$$

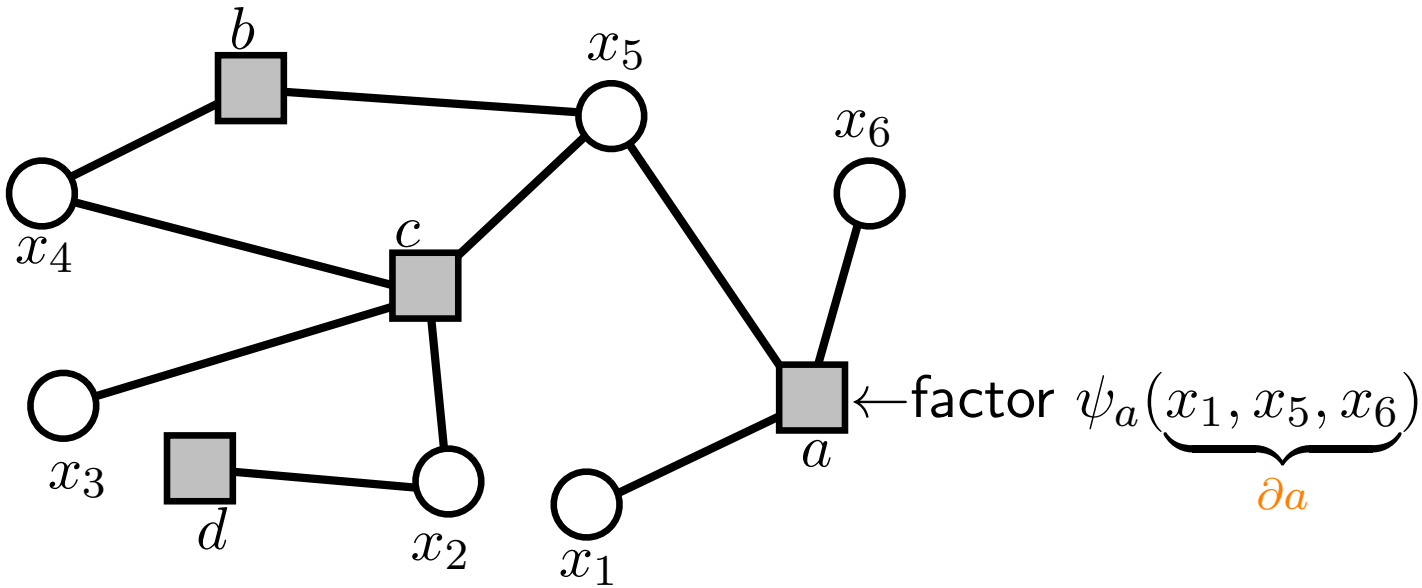
Max-product algorithm

- **max-product algorithm** computes **max-marginal**:

$$\tilde{\mu}_i(x_i) \quad \equiv \quad \max_{x_{V \setminus \{i\}}} \mu(x)$$

- when each max-marginal is uniquely achieved, that is $x_i^* \in \arg\max_{x_i} \tilde{\mu}_i(x_i)$ is unique for all $i \in V$, then $x^* = (x_1^*, \ldots, x_n^*)$ is the global MAP configuration
- when there is a tie, we can keep track of 'backpointers' on what maximizing value of the neighbors are to recover a global MAP configuration
- note that in general $\arg\max_{x_i} \tilde{\mu}_i(x_i) \neq \arg\max_{x_i} \mu(x_i)$

Max-product algorithm for factor graphs



$$\mu(x) = \frac{1}{Z} \prod_{a \in F} \psi_a(x_{\partial a})$$

- most generic form of max-product algorithm on a Factor Graph
  - message update:

$$\nu_{i\to a}^{(t+1)}(x_i) = \prod_{b\in\partial i\setminus\{a\}} \tilde{\nu}_{b\to i}^{(t)}(x_i)$$

$$\tilde{\nu}_{a\to i}^{(t)}(x_i) = \max_{x_{\partial a\setminus\{i\}}} \prod_{j\in\partial a\setminus\{i\}} \nu_{j\to a}^{(t)}(x_j)\psi_a(x_{\partial a})$$

  - backtracking:

$$\delta_{a\to i}^{(t)}(x_i) = \arg\max_{x_{\partial a\setminus\{i\}}} \prod_{j\in\partial a\setminus\{i\}} \nu_{j\to a}^{(t)}(x_j)\psi_a(x_{\partial a})$$

  - max-marginal:

$$\tilde{\mu}_i^{(t)}(x_i) = \prod_{a\in\partial i} \tilde{\nu}_{a\to i}^{(t)}(x_i)$$

- for pairwise MRFs there are two ways to reduce this updates into single update of single messages:

$$\nu_{i\to j}^{(t+1)}(x_i) = \prod_{k\in\partial i\setminus\{j\}} \left\{ \max_{x_k} \psi(x_k, x_i)\nu_{k\to i}^{(t)}(x_k) \right\} \qquad \tilde{\nu}_{i\to j}^{(t+1)}(x_j) = \max_{x_i} \psi(x_i, x_j) \prod_{k\in\partial i\setminus\{j\}} \tilde{\nu}_{k\to i}^{(t)}(x_i)$$

$$\delta_{i\to j}^{(t+1)}(x_i) = \arg\max_{x_{\partial i\setminus\{j\}}} \prod_{k\in\partial i\setminus\{j\}} \psi(x_k, x_i)\nu_{k\to i}^{(t)}(x_k) \qquad \delta_{i\to j}^{(t+1)}(x_j) = \arg\max_{x_i} \psi(x_i, x_j) \prod_{k\in\partial i\setminus\{j\}} \tilde{\nu}_{k\to i}^{(t)}(x_i)$$

- **(parallel) max-product algorithm for pairwise MRFs**
  1. initialize $\nu_{i\to j}^{(0)}(x_i) = 1/|\mathcal{X}|$ for all $(i,j) \in D$
  2. for $t \in \{0, 1, \ldots, t_{\max}\}$
     for all $(i,j) \in D$ compute

$$\nu_{i\to j}^{(t+1)}(x_i) = \prod_{k\in\partial i\setminus j} \left\{ \max_{x_k} \psi_{ik}(x_i,x_k)\nu_{k\to i}^{(t)}(x_k) \right\}$$

$$\delta_{i\to j}^{(t+1)}(x_i) = \arg\max_{x_{\partial i\setminus j}} \prod_{k\in\partial i\setminus j} \left\{ \psi_{ik}(x_i,x_k)\nu_{k\to i}^{(t)}(x_k) \right\}$$

  3. for each $i \in V$ compute max-marginal

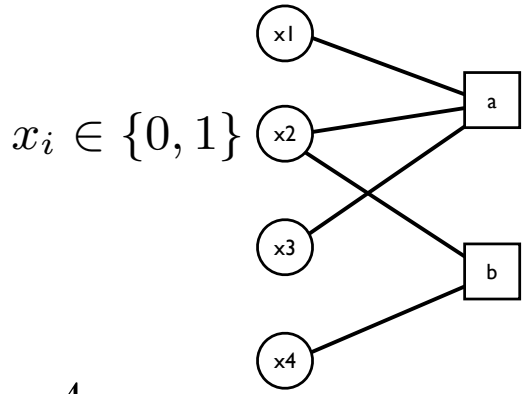
$$\tilde{\mu}_i(x_i) = \prod_{k\in\partial i} \left\{ \max_{x_k} \psi_{ik}(x_i,x_k)\nu_{k\to i}^{(t_{\max}+1)}(x_k) \right\}$$

$$\delta_i(x_i) = \arg\max_{x_{\partial i}} \prod_{k\in\partial i} \left\{ \psi_{ik}(x_i,x_k)\nu_{k\to i}^{(t_{\max}+1)}(x_k) \right\}$$

  4. backtrack to find a MAP configuration
- related to dynamic programming
- exact when $t_{\max} \geq \mathrm{diam}(T)$
- computes max-marginals in $O(n|\mathcal{X}|^2 \cdot \mathrm{diam}(T))$ operations

Example: decoding LDPC codes

- LDPC code is defined by a factor graph model



  - block length $n = 4$
  - number of factors $m = 2$
  - allowed messages $= \{0000, 0111, 1010, 1101\}$

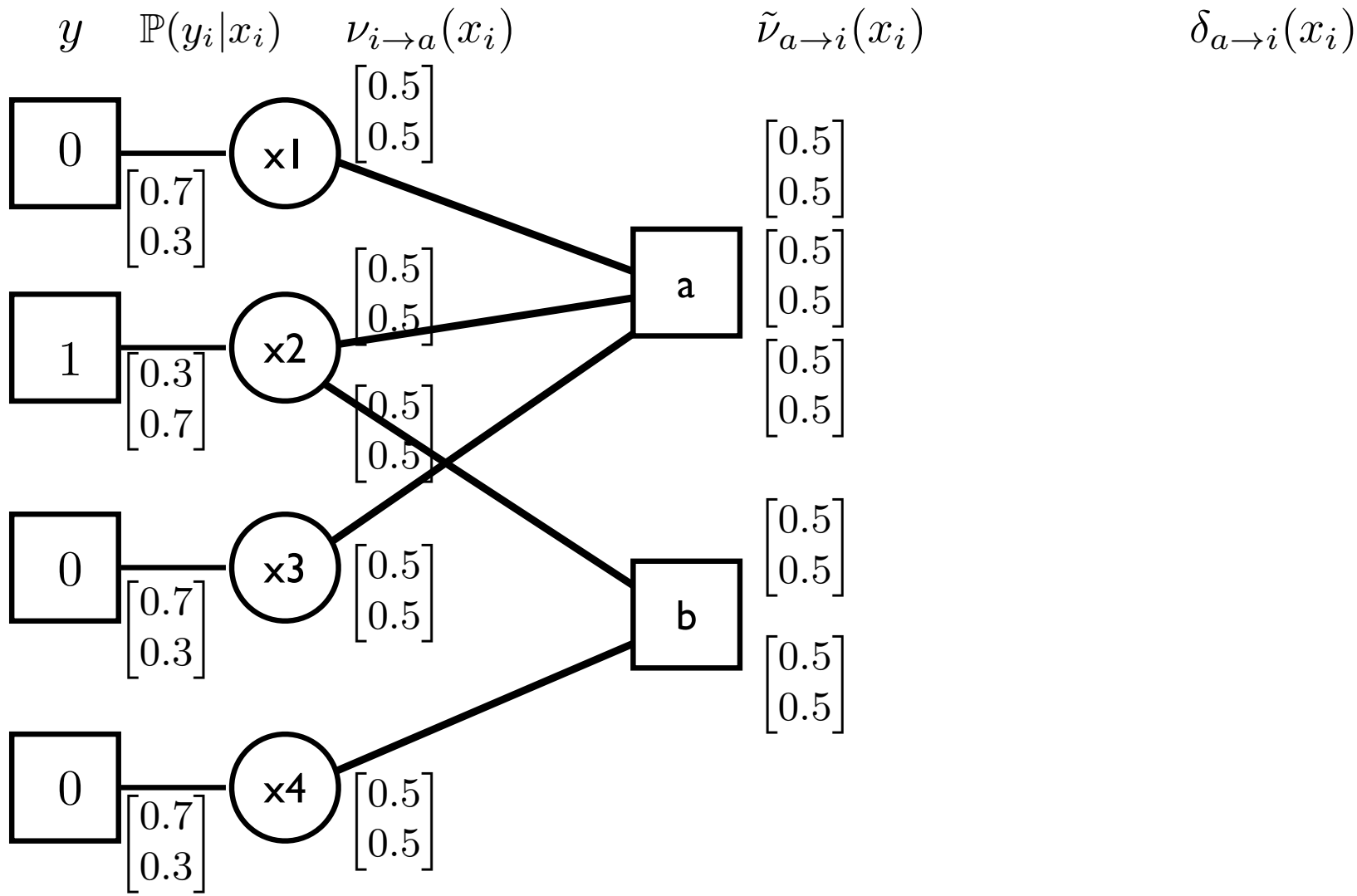
- decoding using max-product algorithm (for BSC with $\epsilon = 0.3$)

$$\mu_y(x) = \frac{1}{Z} \prod_{i \in V} \mathbb{P}_{Y|X}(y_i|x_i) \prod_{a \in F} \mathbb{I}(\oplus x_{\partial a} = 0)$$

- use (parallel) max-product algorithm to find MAP estimate:

$$\hat{x} \;=\; \arg\max \mu(x)$$

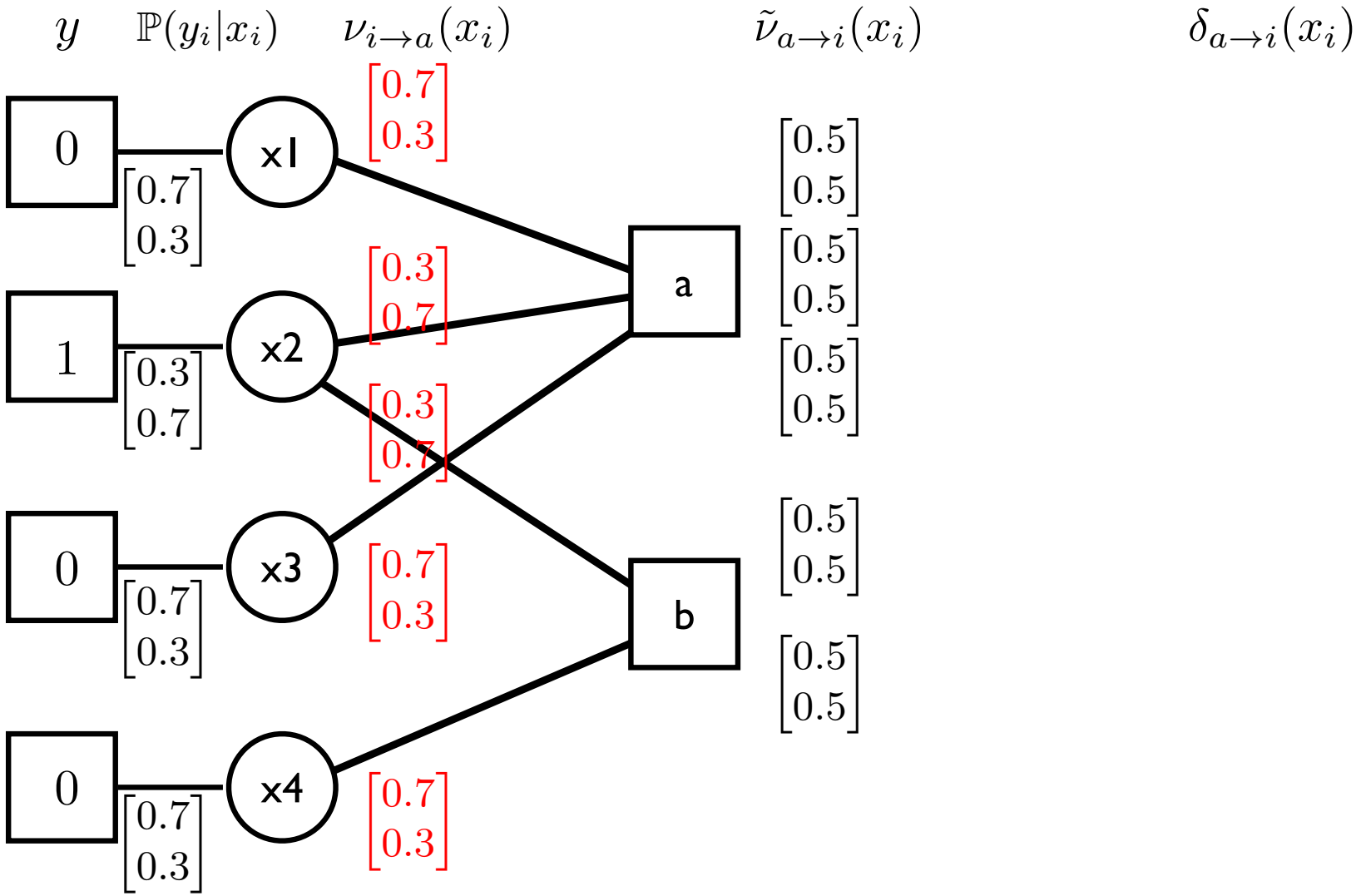
  - minimizes block error rate



$$\nu_{i\to a}^{(t+1)}(x_i) = \mathbb{P}(y_i|x_i) \prod_{b\in\partial i\setminus\{a\}} \tilde{\nu}_{b\to i}^{(t)}(x_i)$$

$$\tilde{\nu}_{a\to i}^{(t+1)}(x_i) = \max_{x_{\partial a\setminus\{i\}}} \prod_{j\in\partial a\setminus\{i\}} \nu_{j\to a}(x_j)\mathbb{I}(\oplus x_{\partial a} = 0)$$

$$\delta_{a\to i}^{(t)}(x_i) = \arg\max_{x_{\partial a\setminus\{i\}}} \prod_{j\in\partial a\setminus\{i\}} \nu_{j\to a}^{(t)}(x_j)\mathbb{I}(\oplus x_{\partial a} = 0)$$



$$\nu_{i\to a}^{(t+1)}(x_i) = \mathbb{P}(y_i|x_i) \prod_{b\in\partial i\setminus\{a\}} \tilde{\nu}_{b\to i}^{(t)}(x_i)$$

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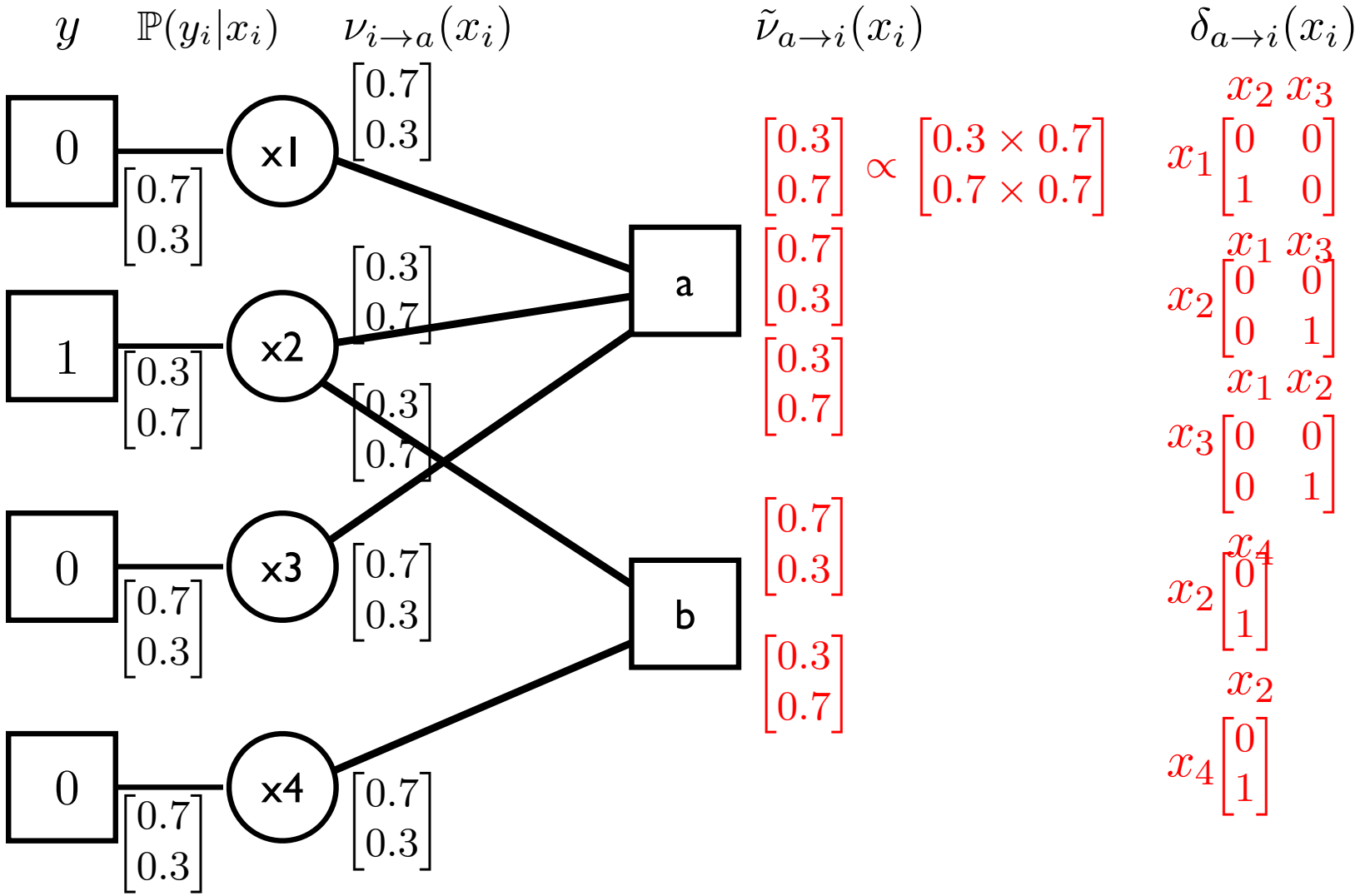
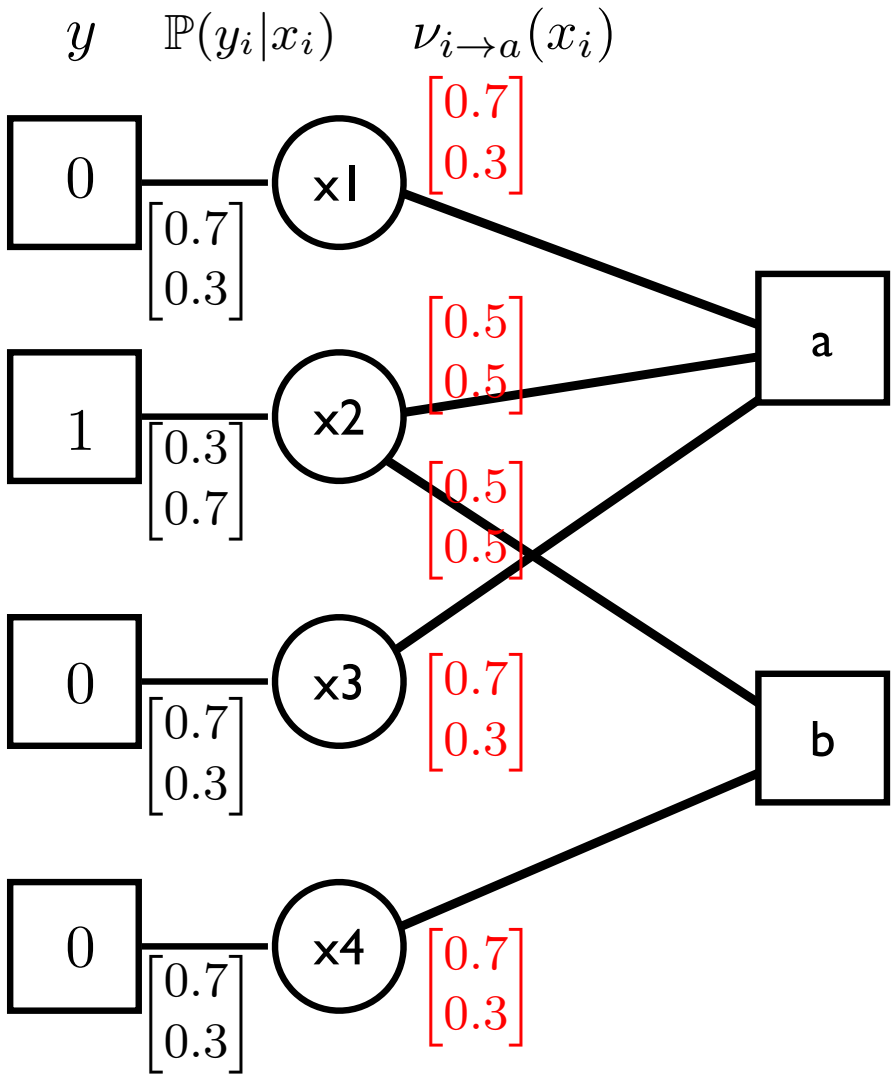
$$\delta_{a\to i}^{(t)}(x_i) = \arg\max_{x_{\partial a\setminus\{i\}}} \prod_{j\in\partial a\setminus\{i\}} \nu_{j\to a}^{(t)}(x_j)\mathbb{I}(\oplus x_{\partial a} = 0)$$



$$\nu_{i\to a}^{(t+1)}(x_i) = \mathbb{P}(y_i|x_i)\prod_{b\in\partial i\setminus\{a\}}\tilde{\nu}_{b\to i}^{(t)}(x_i)$$

$$\tilde{\nu}_{a\to i}^{(t+1)}(x_i) = \max_{x_{\partial a\setminus\{i\}}}\prod_{j\in\partial a\setminus\{i\}}\nu_{j\to a}(x_j)\mathbb{I}(\oplus x_{\partial a}=0)$$

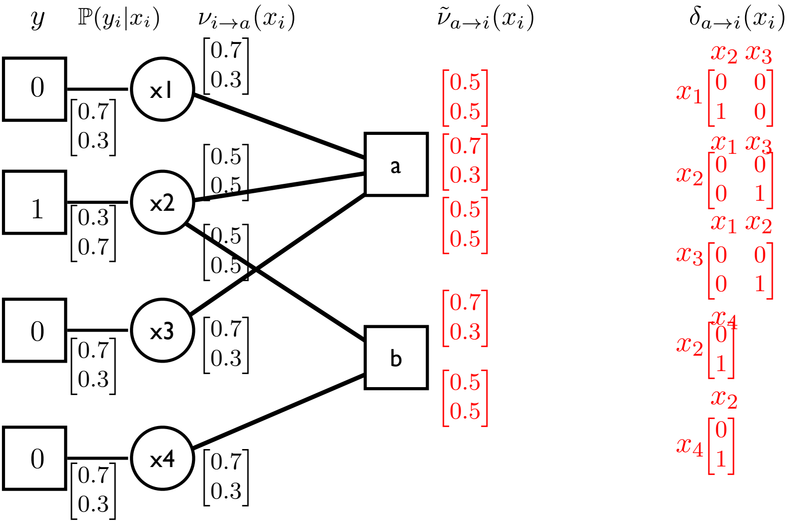
$$\delta_{a\to i}^{(t)}(x_i) = \arg\max_{x_{\partial a\setminus\{i\}}}\prod_{j\in\partial a\setminus\{i\}}\nu_{j\to a}^{(t)}(x_j)\mathbb{I}(\oplus x_{\partial a}=0)$$



$$\nu_{i\to a}^{(t+1)}(x_i) = \mathbb{P}(y_i|x_i)\prod_{b\in\partial i\setminus\{a\}}\tilde{\nu}_{b\to i}^{(t)}(x_i)$$

$$\tilde{\nu}_{a\to i}^{(t+1)}(x_i) = \max_{x_{\partial a\setminus\{i\}}}\prod_{j\in\partial a\setminus\{i\}}\nu_{j\to a}(x_j)\mathbb{I}(\oplus x_{\partial a}=0)$$

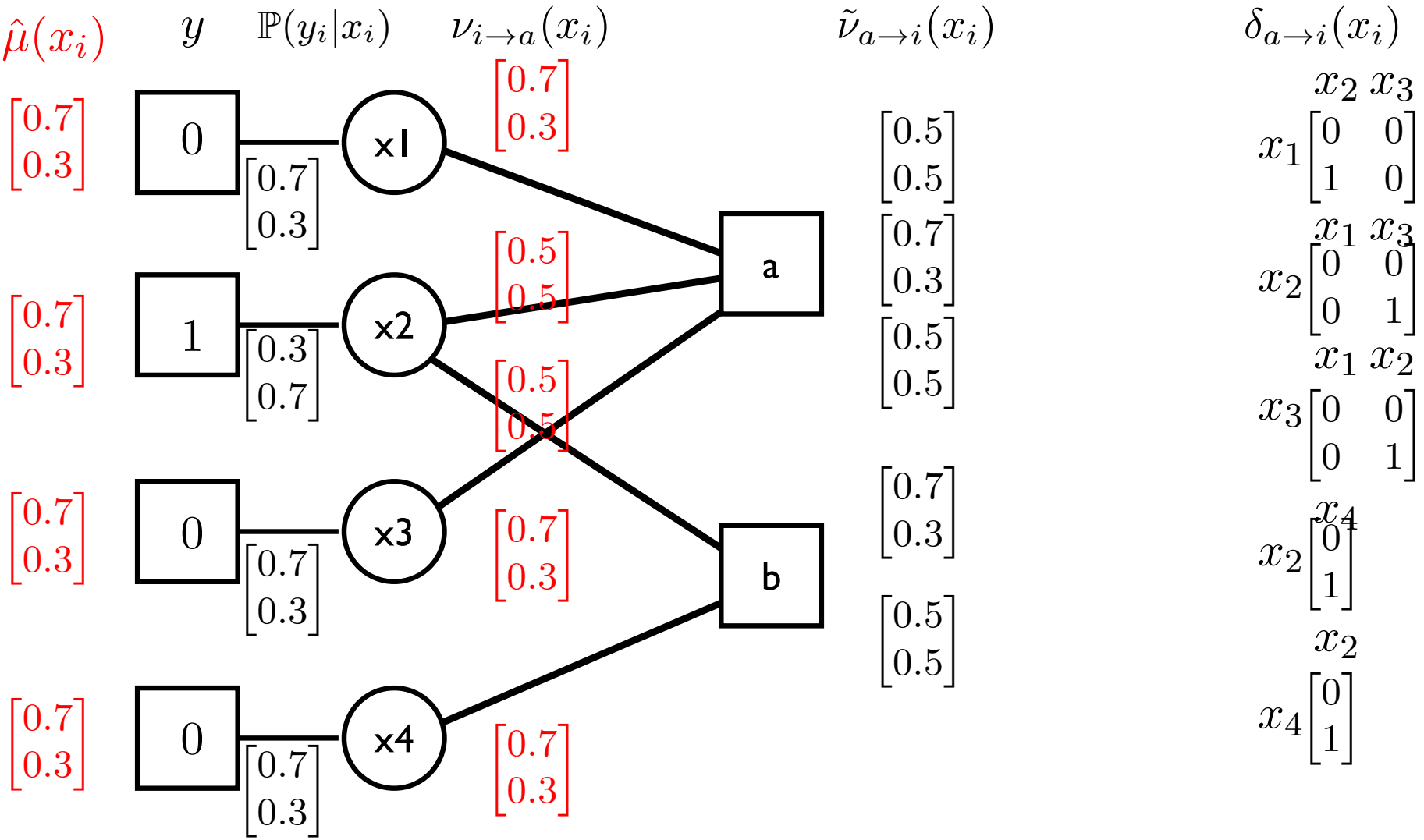
$$\delta_{a\to i}^{(t)}(x_i) = \arg\max_{x_{\partial a\setminus\{i\}}}\prod_{j\in\partial a\setminus\{i\}}\nu_{j\to a}^{(t)}(x_j)\mathbb{I}(\oplus x_{\partial a}=0)$$



$$\nu^{(t+1)}_{i\to a}(x_i) = \mathbb{P}(y_i|x_i)\prod_{b\in\partial i\setminus\{a\}} \tilde{\nu}^{(t)}_{b\to i}(x_i)$$

$$\tilde{\nu}^{(t+1)}_{a\to i}(x_i) = \max_{x_{\partial a\setminus\{i\}}} \prod_{j\in\partial a\setminus\{i\}} \nu_{j\to a}(x_j)\mathbb{I}(\oplus x_{\partial a} = 0)$$

$$\delta^{(t)}_{a\to i}(x_i) = \arg\max_{x_{\partial a\setminus\{i\}}} \prod_{j\in\partial a\setminus\{i\}} \nu^{(t)}_{j\to a}(x_j)\mathbb{I}(\oplus x_{\partial a} = 0)$$



$$\nu_{i\to a}^{(t+1)}(x_i) = \mathbb{P}(y_i|x_i) \prod_{b\in\partial i\setminus\{a\}} \tilde{\nu}_{b\to i}^{(t)}(x_i)$$

$$\tilde{\nu}_{a\to i}^{(t+1)}(x_i) = \max_{x_{\partial a\setminus\{i\}}} \prod_{j\in\partial a\setminus\{i\}} \nu_{j\to a}(x_j)\mathbb{I}(\oplus x_{\partial a} = 0)$$

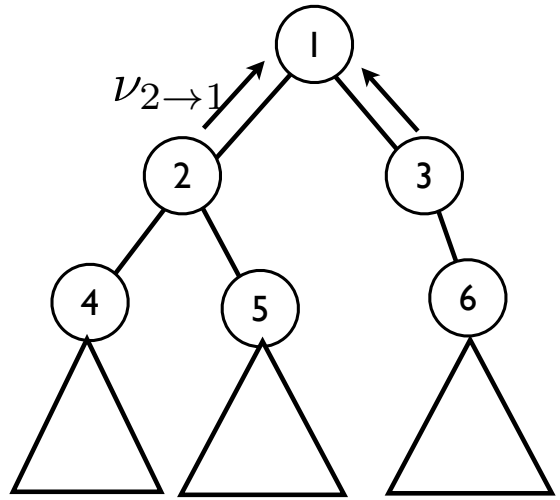
$$\delta_{a\to i}^{(t)}(x_i) = \arg\max_{x_{\partial a\setminus\{i\}}} \prod_{j\in\partial a\setminus\{i\}} \nu_{j\to a}^{(t)}(x_j)\mathbb{I}(\oplus x_{\partial a} = 0)$$

Numerical stability and min-sum algorithm

- multiplying many compatibility functions can lead to numerical issues (e.g. when the values are close to zero)
- one remedy is to work in log domain
- **min-sum algorithm**

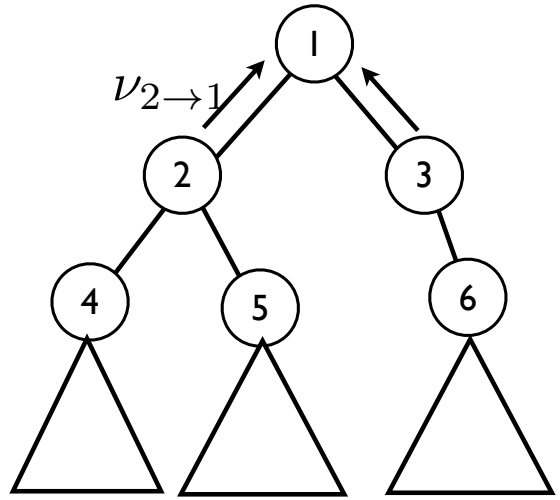
$$\nu_{i\to j}(x_i) = \sum_{k\in\partial i\setminus j} \min_{x_k} \Big\{ -\log\psi_{ki}(x_k,x_i) + \nu_{k\to i}(x_k) \Big\}$$

$$\tilde{\mu}_i(x_i) = \exp\Big\{ \sum_{k\in\partial i} \min_{x_k} \Big\{ -\log\psi_{ki}(x_k,x_i) + \nu_{k\to i}(x_k) \Big\} \Big\}$$



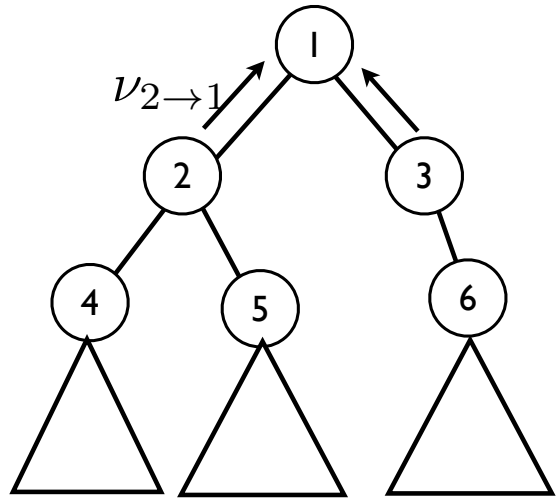
- define graphical model on sub-trees

$$
\begin{aligned}
T_{2\to 1} = (V_{2\to 1}, E_{2\to 1}) &\equiv \text{Subtree rooted at 2 and excluding 1} \\
\mu_{2\to 1}(x_{V_{2\to 1}}) &\equiv \frac{1}{Z(T_{2\to 1})} \prod_{(u,v)\in E_{2\to 1}} \psi(x_u, x_v) \\
\nu_{2\to 1}(x_2) &\equiv \max_{x_{V_{2\to 1}\setminus\{2\}}} \mu_{2\to 1}(x_{V_{2\to 1}})
\end{aligned}
$$



- the messages from neighbors are sufficient to compute the max-marginal $\tilde{\mu}(x_1)$

$$
\begin{aligned}
\tilde{\mu}(x_1) \quad &\propto \quad \max_{x_{V\setminus\{1\}}} \prod_{(u,v)\in E} \psi(x_u, x_v) \\
&= \quad \max_{x_{V_{2\to 1}}, x_{V_{3\to 1}}} \psi(x_1,x_2)\psi(x_1,x_3)\Big\{ \prod_{(u,v)\in E_{2\to 1}} \psi(x_u,x_v)\Big\}\Big\{ \prod_{(u,v)\in E_{3\to 1}} \psi(x_u,x_v)\Big\} \\
&= \quad \max_{x_{V_{2\to 1}}, x_{V_{3\to 1}}} \prod_{k\in\partial 1} \Big\{\psi(x_1,x_k)\Big\{ \prod_{(u,v)\in E_{k\to 1}} \psi(x_u,x_v)\Big\}\Big\} \\
&= \quad \prod_{k\in\partial 1} \max_{x_{V_{k\to 1}}} \Big\{\psi(x_1,x_k)\Big\{ \prod_{(u,v)\in E_{k\to 1}} \psi(x_u,x_v)\Big\}\Big\} \\
&= \quad \prod_{k\in\partial 1} \max_{x_k} \Big\{\psi(x_1,x_k)\underbrace{\Big\{ \max_{x_{V_{k\to 1}\setminus\{k\}}} \prod_{(u,v)\in E_{k\to 1}} \psi(x_u,x_v)\Big\}}_{\nu_{k\to 1}(x_k)}\Big\}
\end{aligned}
$$



- recursion on sub-trees to compute the messages ν

$$\begin{aligned}
\mu_{2\to 1}(x_{V_{2\to 1}}) &= \frac{1}{Z(T_{2\to 1})} \prod_{(u,v)\in E_{2\to 1}} \psi(x_u, x_v) \\
&\propto \psi(x_2, x_4)\psi(x_2, x_5)\Big\{ \prod_{(u,v)\in E_{4\to 2}} \psi(x_u, x_v)\Big\}\Big\{ \prod_{(u,v)\in E_{5\to 2}} \psi(x_u, x_v)\Big\} \\
&\propto \psi(x_2, x_4)\, \psi(x_2, x_5)\mu_{4\to 2}(x_{V_{4\to 2}})\mu_{5\to 2}(x_{V_{5\to 2}})
\end{aligned}$$

$$
\begin{aligned}
\nu_{2\to 1}(x_2) &= \max_{x_{V_{2\to 1}\setminus 2}} \mu_{2\to 1}(x_{V_{2\to 1}}) \\
&\propto \max_{x_{V_{2\to 1}\setminus 2}} \psi(x_2,x_4)\,\psi(x_2,x_5)\mu_{4\to 2}(x_{V_{4\to 2}})\mu_{5\to 2}(x_{V_{5\to 2}}) \\
&\propto \Big\{\max_{x_{V_{4\to 2}}} \psi(x_2,x_4)\mu_{4\to 2}(x_{V_{4\to 2}})\Big\}\Big\{\max_{x_{V_{5\to 2}}} \psi(x_2,x_5)\mu_{5\to 2}(x_{V_{5\to 2}})\Big\} \\
&= \Big\{\max_{x_4}\psi(x_2,x_4)\max_{x_{V_{4\to 2}\setminus\{4\}}} \mu_{4\to 2}(x_{V_{4\to 2}})\Big\}\Big\{\max_{x_5}\psi(x_2,x_5)\max_{x_{V_{5\to 2}\setminus\{5\}}} \mu_{5\to 2}(x_{V_{5\to 2}})\Big\} \\
&\propto \Big\{\max_{x_4}\psi(x_2,x_4)\nu_{4\to 2}(x_4)\Big\}\Big\{\max_{x_5}\psi(x_2,x_5)\nu_{5\to 2}(x_5)\Big\}
\end{aligned}
$$

- update messages via recursion with uniform initialization $\nu_{i\to j}(x_i) = \frac{1}{|\mathcal{X}|}$ for all leaves i

$$
\nu_{2\to 1}(x_2) = \prod_{k\in\partial 2\setminus\{1\}} \Big\{\max_{x_k}\psi_{2k}(x_2,x_k)\nu_{k\to 2}(x_k)\Big\}
$$

- to obtain a global configuration, we need to store the $\arg\max$ for each message for backtracking

$$
\delta_{2\to 1}(x_2) = \arg\max_{x_{\partial 2\setminus\{1\}}} \prod_{k\in\partial 2\setminus 1} \Big\{\psi_{2k}(x_2,x_k)\nu_{k\to 2}(x_k)\Big\}
$$